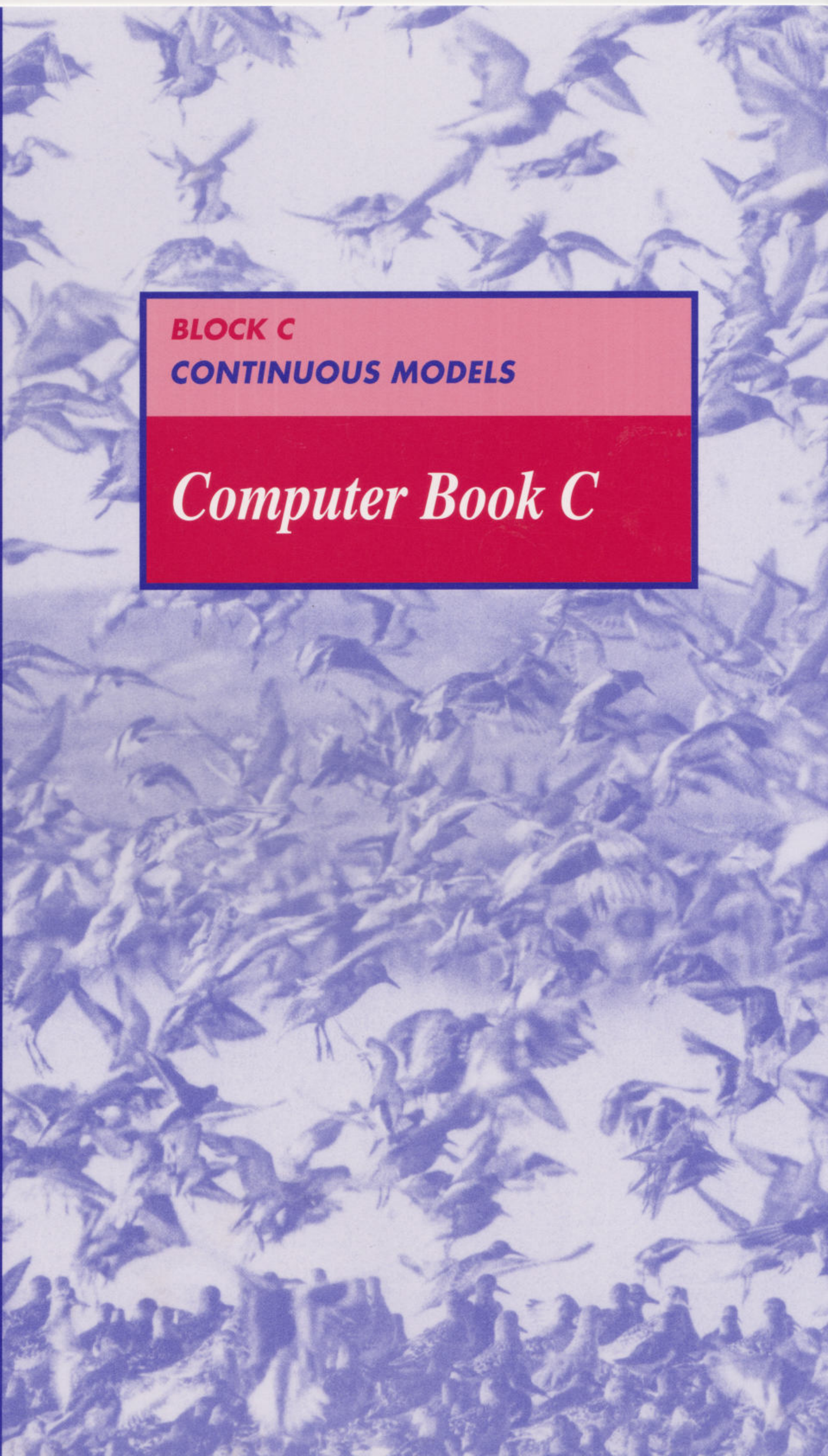
MST121 CB C

The Open University

A first level interdisciplinary course

Using Mathematics



BLOCK C

CONTINUOUS MODELS

Computer Book C

COMPUTER BOOK

C

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Prepared by the course team

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About this course

This computer book forms part of the course MST121 *Using Mathematics*. This course and the courses MU120 *Open Mathematics* and MS221 *Exploring Mathematics* provide a flexible means of entry to university-level mathematics. Further details may be obtained from the address below.

MST121 uses the software program Mathcad (MathSoft, Inc.) and other software to investigate mathematical and statistical concepts and as a tool in problem solving. This software is provided as part of the course.

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Contents

Guidance notes

This computer book contains those sections of the chapters in Block C which require you to use Mathcad. Each of these chapters contains instructions as to when you should first refer to particular material in this computer book, so you are advised not to work on the activities here until you have reached the appropriate points in the chapters.

In order to use this computer book, you will need the following Mathcad files.

Chapter C1

121C1-01 Differentiation

Chapter C2

121C2-01 Integration as a limit of summations (Optional)

Chapter C3

121C3-01 Direction fields and solution curves
121C3-02 Euler's method

Instructions for installing these files onto your computer's hard disk, and for opening them, are given in Chapter A0.

The computer activities for Chapters C1 and C2 require you also to work with Mathcad worksheets which you have created yourself.

Activities based on software vary both in nature and in length. Sometimes the instructions for an activity appear only in the computer book; in other cases, instructions are given in the computer book and on screen. Feedback on an activity is sometimes provided on screen and is sometimes given in the computer book.

For advice on how each computer session fits into the suggested study patterns, refer to the Study guides in the chapters.

Chapter C1, Section 5
Optimisation with the computer

The Mathcad activities for Chapter C1 have only one prepared file associated with them. This file explains how to differentiate with the computer, which is the topic of Subsection 5.1. To solve optimisation problems, in Subsection 5.2, you will be given guidance on how to create your own Mathcad worksheets.

By the end of this section, you should be able to use Mathcad to carry out the process of differentiation and to solve optimisation problems, and also be a more skilled and independent Mathcad user.

5.1 Finding derivatives

In Mathcad, derivatives are found using the $\frac{d}{dx}$ operator. This operator can be used both to find a general formula for the derivative of a function f, and to find the numerical value of the derivative of f at a particular point.

For example, the derivative of the function $f(x) = \frac{1}{5}x^2$ can be described by the formula $f'(x) = \frac{2}{5}x$, which holds for all real values of x. Mathcad can obtain this expression for the derivative by using the $\frac{d}{dx}$ operator and its symbolic commands, as described in Activity 5.1. By defining a value for x before these calculations, $x = 3$ say, the value of the derivative at that particular point can be found as $f'(3) = \frac{2}{5} \times 3 = 1.2$, as you will see in Activity 5.2.

Activity 5.1 Finding a formula for the derivative

Open Mathcad file **121C1-01 Differentiation**. Page 1 introduces the worksheet. Work through page 2, and then carry out Task 1 on page 3.

Remember to create your own working copy of the file.

Solutions are given on page 29.

Comment

- ◇ Each part of Task 1 concerns a function that you were asked to differentiate in the main text. In the solutions, where the expression obtained using Mathcad differs in appearance from that obtained by hand, both expressions are given. If you cannot see immediately why such a pair of expressions are equivalent, it is a good idea to copy down one expression and use algebra to verify that it can be rearranged to give the other.
- ◇ Make sure that the variables entered in the two placeholders of the $\frac{d}{dx}$ operator match. For example, if you mistakenly try to evaluate symbolically the expression

  $$\frac{d}{dt}\cos(4x),$$

  then you will obtain the answer 0.
- ◇ Brackets can be crucial when entering an expression into the right-hand placeholder of the $\frac{d}{dx}$ operator. For example, if the expression $x^3 - 6x^2 - 15x + 54$ is entered without being enclosed in brackets, Mathcad will differentiate the x^3 term but not the others.

An example of such a quotient is

$$\frac{\sin(t) - t^2}{e^t}.$$

You may find that Mathcad will not allow you to enter into the placeholder a quotient whose numerator is a sum of terms (it depends on how you enter the quotient). This difficulty can be avoided by enclosing the quotient in brackets.

To avoid such problems, it is good practice to include outer brackets around expressions that you enter into the right-hand placeholder of the $\frac{d}{dx}$ operator. This also helps to make the expression to be differentiated clear on the screen.

These symbolic keywords were introduced in Chapter A0, file 121A0-05. See also *A Guide to Mathcad* for further details.

◇ Sometimes the expression for a derivative obtained by Mathcad can be 'improved' by simplifying it. In place of symbolic evaluation ($\rightarrow$), either of the symbolic keywords 'simplify' and 'factor' can be applied to obtain a derivative. The outcome from these may or may not be the same as that from using $\rightarrow$, but will sometimes be in a more convenient form. However, what Mathcad regards as 'simpler' may not necessarily seem so to a human observer!

Mathcad notes

Remember that Mathcad notes are *optional*.

The expressions e^x and $\exp(x)$ are equivalent in Mathcad, but the latter form is always used in the output of symbolic calculations, irrespective of the form used for input. Similarly, the power $\frac{1}{2}$ appears in output rather than the square root sign.

Activity 5.1 showed how to use use the $\frac{d}{dx}$ operator and symbolic evaluation to obtain an algebraic expression for the derivative. This replicates what you might do by hand, but Mathcad can also be used to differentiate functions that would be rather complicated to do by hand. We turn next to how Mathcad can be used to find the numerical value for the derivative at a particular point.

Activity 5.2 Evaluating the derivative at a point

You should still be working with Mathcad file 121C1-01.

Turn to page 4 of the worksheet, and carry out Task 2.

Solutions are given further down page 4 of the worksheet.

Comment

◇ Note the comments made towards the bottom of page 4 of the worksheet. Once a value has been defined for the differentiation variable (x, say), then the $\frac{d}{dx}$ operator can be evaluated either symbolically ($\frac{d}{dx}(\ldots) \rightarrow$) or numerically ($\frac{d}{dx}(\ldots) =$) to find the numerical value of the derivative at that particular value of x.

These two evaluation methods usually give the same numerical value, but Mathcad calculates the two results in different ways. For symbolic evaluation, Mathcad finds a general formula for the derivative and then evaluates this formula for the particular value of x, whereas for numerical evaluation, Mathcad uses a numerical algorithm to find an approximate decimal value.

Essentially, this involves a direct application of the definition of derivative, as given by equation (1.3) in Chapter C1. See also the second Mathcad note below.

◇ If the expression $\frac{d}{dx}f(x)$ is entered, where a definition for the function $f(x)$ is provided earlier in the worksheet, then Mathcad can find either a symbolic derivative for $f(x)$ or the numerical value of this derivative at any specified point, just as before.

Mathcad notes

◇ When you enter $\rightarrow$ to evaluate an expression symbolically, or $=$ to evaluate it numerically, it doesn't matter where on the expression the blue editing lines are. All that matters is that the expression is complete, with every placeholder filled in. These two evaluation methods also behave in a similar way if a change is made to the worksheet, above or to the left of a calculation. In automatic calculation mode (the default), the result of a calculation involving either $\rightarrow$ or $=$ is updated automatically, while in manual mode, you can press the `[F9]` key to update the result.

To change from automatic to manual calculation mode, or vice versa, click on **Automatic Calculation** in the **Math** menu.

◇ When evaluating a derivative numerically ($\frac{d}{dx}(\ldots) =$), you must define earlier in the worksheet the point at which the derivative is to be found; for example, $x := 3$. Mathcad then uses a numerical algorithm to obtain an approximation to the exact value of the derivative at that point, which is usually accurate to 7 or 8 significant figures. Very occasionally the method fails, in which case the derivative is highlighted in red. Clicking on this expression reveals the error message 'Can't converge to a solution.'.

Without this prior definition, x appears in red in the $\frac{d}{dx}$ operator, as an undefined variable.

Now close file 121C1-01.

Sometimes you may want to use Mathcad to find a second-order derivative. One way to do this is by using the $\frac{d}{dx}$ operator twice. You can enter the $\frac{d}{dx}$ operator in your worksheet, then enter the $\frac{d}{dx}$ operator again into the right-hand placeholder, and then fill in all the placeholders appropriately. For example, Mathcad gives the result

$$\frac{d}{dx}\left(\frac{d}{dx}(x^3 - 6x^2 - 15x + 54)\right) \rightarrow 6x - 12.$$

Another way to find a second-order derivative in Mathcad is to use the $\frac{d^n}{dx^n}$ operator, for which there is a button on the 'Calculus' toolbar. You should enter '2' in the bottom placeholder (which will cause a 2 to appear in the top placeholder as well), and fill in the other placeholders just as for the $\frac{d}{dx}$ operator. You can evaluate the resulting expression either symbolically or numerically, in the same way as for expressions involving the $\frac{d}{dx}$ operator. For example, Mathcad gives the result

The keyboard alternative is `[Ctrl]?` (given by the three keys `[Ctrl]`, `[Shift]` and `/`).

$$\frac{d^2}{dx^2}(x^3 - 6x^2 - 15x + 54) \rightarrow 6x - 12.$$

5.2 Optimisation

Optimisation involves finding the greatest or least value taken by a function on an interval. In the main text you saw how to apply the Optimisation Procedure by hand, but all the calculations required can also be performed using Mathcad. In some cases, Mathcad can be applied to check calculations already done by hand, while in other cases, calculations can be carried out that are too complicated to do by hand.

See Chapter C1, Subsection 2.3.

In this subsection you will solve two optimisation problems, the first a minimisation problem and the second a maximisation problem.

The orienteer's problem

See Computer Book A, Chapter A3, Subsection 5.5.

The orienteer's problem was described in Computer Book A. The orienteer starts at a point in a forest and needs to reach a final point on a path. The problem is to find where the orienteer should aim to join the path, where the joining point is x km from a fixed point O on the path, in order to minimise the time taken overall. For the particular data given, this involves finding the value of x that gives the least value of the journey time (in hours)

$$f(x) = 0.125\sqrt{1+x^2} + 0.0625(2-x) \quad (x \text{ in } [0,2]).$$

The graph of $y = f(x)$ is shown in Figure 5.1.

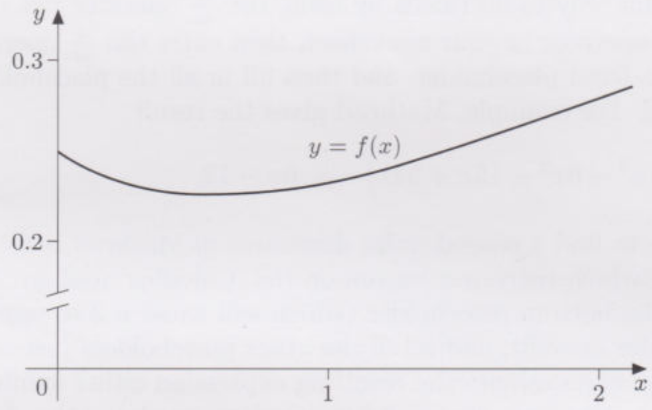


Figure 5.1 Graph of the function for the orienteer's problem

In Computer Book A, you found an approximate solution to this problem using Mathcad, by zooming in on the graph and applying the graph trace tool. You are now in a position to solve the problem more accurately, using differentiation.

In Activity 5.3 you are asked to use the Optimisation Procedure to solve the orienteer's problem. In place of a prepared Mathcad file for this activity, you are guided through creating your own Mathcad worksheet. This includes a requirement to enter some explanatory text, to make the worksheet more comprehensible.

The instructions for this activity may look lengthy, but several of them describe general Mathcad techniques that can save you time and effort, now and in the future. The instructions are given mostly in a mouse/click way, but details of keyboard alternatives are provided throughout. If you wish to use the toolbar buttons, then the activity makes use of buttons on the 'Calculator', 'Calculus', 'Boolean' and 'Symbolic' toolbars.

To open a toolbar, either click on the appropriate button on the 'Math' toolbar, or use the **View** menu, **Toolbars**.

Activity 5.3 Minimising the orienteer's journey time

The following instructions guide you through creating a Mathcad worksheet to carry out the Optimisation Procedure, in order to solve the orienteer's problem. Part (b) describes how to enter a title in the worksheet. You should similarly enter any other text that you think is appropriate. For example, you could include a line of text before each step of the Optimisation Procedure, to explain what you are doing.

A Guide to Mathcad contains detailed information on creating and editing your own worksheets. See also Activity 2.3 in Chapter A0.

(a) Begin by creating a new worksheet, as follows. Select the **File** menu and choose **New...** . In the list of templates that appears, **Normal** should be selected by default. If not, click on it. Then click on the **OK** button to create a new (Normal) worksheet. (Alternatively, type `[Ctrl]n`, or click on the 'New' button on the standard toolbar.)

If you have just started Mathcad running, then there is no need to do this, as it automatically starts with a new (Normal) worksheet.

(b) Enter a title at the top of your worksheet. To do this, click to position the red cross cursor in an appropriate place, and choose **Text Region** from the **Insert** menu. (Alternatively, type a double-quote `"`, given by `[Shift]2`.) Then type a suitable title, for example, `Optimisation - the orienteer's problem`, in the text box. To finish, click anywhere outside the text box or press `[Ctrl][Shift][Enter]`. If you need to edit the text later, simply click on it.

Any text that you enter can later be moved, or deleted, as you wish.

(c) Enter a definition of the function to be optimised, which is

$$f(x) = 0.125\sqrt{1+x^2} + 0.0625(2-x).$$

You can use the keyboard and buttons on the 'Calculator' toolbar to enter this, or just type the key sequence

`f(x):0.125*\1+x^2[Space][Space][Space][Space]+0.0625*(2-x)`

The \ (backslash) keystroke creates the square root sign.

If you have difficulty in following these instructions, then you may like to look ahead to the first item in the Comment below, which shows what should eventually appear on the screen.

(d) Now you are ready to solve the orienteer's problem, by carrying out the three steps of the Optimisation Procedure, as follows.

Step 1: Find the stationary points of f

Using the $\frac{d}{dx}$ button on the 'Calculus' toolbar, enter the expression $\frac{d}{dx}f(x)$. Then click on the 'Equal to' = button on the 'Boolean' toolbar, and enter `0` (zero). (Alternatively, just type `?x[Tab]f(x)[Ctrl]=0`. The `?` (question mark, given by `[Shift]/`) enters the $\frac{d}{dx}$ operator, `[Tab]` moves between the placeholders, and `[Ctrl]=` gives the special equals sign.)

Note that this button gives the special equals sign, which equates the expressions on either side of it. (It has thicker lines than the equals sign obtained from the 'Evaluate Numerically' = button on the 'Calculator' toolbar.)

The stationary points are the points where the derivative of the function f is zero, that is, the solutions of the equation just typed in. Hence (with the vertical blue editing line still at the right-hand end of this equation), click on the 'solve' button on the 'Symbolic' toolbar, then type `x` into the empty placeholder that appears. (Alternatively, just type `[Ctrl][Shift].solve,x`.) Finally, click elsewhere on the page, or press `[Enter]`.

You should see the solution $0.577\,350\ldots$ appear to the right of '$\rightarrow$'. This is the single value of x at which the function f has a stationary point.

Step 2: Evaluate f at each of the relevant points

Use Mathcad to evaluate the original expression $f(x)$ at the two endpoints of the interval, $x = 0$ and $x = 2$, and at the stationary point between them. For example, type `f(0)=` to evaluate $f(0)$. It is sufficient to input the value for the stationary point to three decimal places, $x = 0.577$, or you can use copy and paste to input all 20 digits given in the solution if you wish!

A Comment item below describes in detail how to use the copy and paste facilities.

Step 3: Choose the optimum value

Identify the least of the three function values that you have found. This is the minimum value of $f(x)$ within the given interval.

Remember that the orienteer's problem is a minimisation problem.

You could finish at this point, but it is a good idea to record your conclusions in the worksheet. For example, you could enter the text

```
The least value of f(x) on [0,2] is ?  (at x=?).
```

replacing the question marks with the values that you found. You can add more text if you wish, but the Optimisation Procedure for the orienteer's problem is now completed.

(e) Finally, save your file, by choosing **Save As...** from the **File** menu. You will need to give it a suitable name that indicates the contents, for example, *my121C1-orienteer*. This will help you to organise and identify your files.

A solution to part (d) is given on page 29.

Comment

◇ The Mathcad worksheet should now look something like this:

Your worksheet should contain the calculations shown, but may have different text.

Optimisation - the orienteer's problem

Find the distance x that minimises the time f(x) taken.

$$f(x) := 0.125\sqrt{1+x^2} + 0.0625(2-x)$$

Step 1 Find the stationary points of f

$$\frac{d}{dx}f(x) = 0 \text{ solve}, x \rightarrow .57735026918962576451$$

Step 2 Evaluate f at the endpoints and at the stationary point

$f(0) = 0.25$ $\quad$ $f(2) = 0.28$ $\quad$ $f(0.577) = 0.233$

Step 3 Choose the optimum value

The least value of f(x) on [0, 2] is 0.233 (at x = 0.577).

- ◇ It would have been possible to solve the equation $\frac{d}{dx}f(x) = 0$ as described above but without explicitly entering ' $= 0$ '. When an expression does not contain an equals sign, the symbolic keyword 'solve' will find values of the selected variable for which the expression is equal to zero. (However, it is clearer to read such a line of the worksheet with ' $= 0$ ' included.) Note also that Mathcad omits the zero before the decimal point in the solution.
- ◇ If you want to evaluate $f(0.577\,350\ldots)$ using all 20 decimal places given in the solution for the stationary point, then rather than enter the number from scratch, you can take advantage of Mathcad's copy and paste facilities. To do this, click anywhere in the solution (the horizontal blue editing line extends to select the entire number, regardless of where the vertical editing line is positioned within it) and choose **Copy** from the **Edit** menu. Then type `f()` , and paste this number into the empty placeholder between the brackets by choosing **Paste** from the **Edit** menu. Lastly, click or type `=` to evaluate the function for this value. (The keyboard alternatives for copy and paste are `[Ctrl]c` and `[Ctrl]v`, respectively.)

Copy and paste is a handy way of avoiding the need to retype awkward or lengthy expressions.

Mathcad notes

When working symbolically, a decimal point in the input expression triggers a decimal result, with up to 20 decimal places! However, when working numerically (evaluating an expression using `=`), the number of decimal places displayed depends on the setting for 'Number of decimal places' (**Format** menu, **Result...**, 'Number Format' tab). By default, the results of numerical calculations are displayed to three decimal places.

Now close the Mathcad file that you have created and saved.

A traffic planning model

Traffic planners wish to set up a mathematical model to describe how the volume flow rate of traffic (that is, the number of vehicles which pass a fixed point in a given time) varies with the average velocity of vehicles along a single lane of a road. The eventual purpose of this model is to advise on how traffic flow can be maximised.

The planners assume that each vehicle moves at a constant velocity $v\ \mathrm{m\,s^{-1}}$. On the basis of a subsidiary model and many observations, they estimate that, on average, each driver maintains a distance $v + 0.02v^2$ metres between the front of their vehicle and the back of the vehicle immediately ahead. The average length of a vehicle is estimated to be 5 metres. The model therefore represents the situation as shown in Figure 5.2.

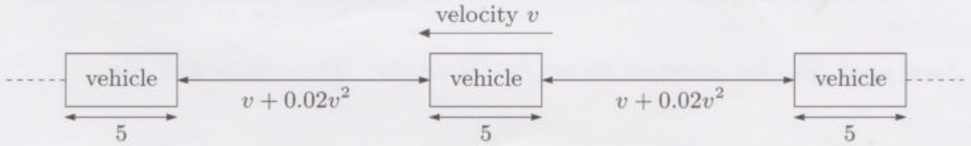


Figure 5.2 Representation of traffic flow

The distance between the fronts of successive vehicles is $5 + v + 0.02v^2$ metres. Each vehicle, travelling at velocity v $\mathrm{m\,s^{-1}}$, covers this distance in $(5 + v + 0.02v^2)/v$ seconds. It follows that the volume flow rate $f(v)$ of traffic (in vehicles per second past a fixed point) is given by

$$f(v) = \frac{v}{5 + v + 0.02v^2}.$$

A velocity of 35 $\mathrm{m\,s^{-1}}$ is about 126 kilometres per hour or 78 miles per hour.

This formula is to apply for $0 \le v \le 35$.

The traffic planners seek the maximum value of $f(v)$ for v in the interval $[0, 35]$. If such a maximum can be found, it can be used to provide an advised speed of travel on the road.

In the next activity you are asked to create a Mathcad worksheet once more, to find both the greatest value of the volume flow rate function $f(v)$ and the value of v for which this occurs.

Activity 5.4 Maximising the volume flow rate of traffic

(a) Create a new (Normal) worksheet, as in Activity 5.3(a). (Alternatively, you may prefer to work with the file that you created for the orienteer's problem in Activity 5.3. To do this, open the file, make a working copy of it using a *different* filename, for example, *my121C1-traffic*, then edit the text and expressions in the worksheet.)

(b) Enter a title at the top of your worksheet, for example, `Maximising the volume flow rate of traffic`.

(c) Solve the volume flow rate problem by finding the maximum value of the function

$$f(v) = \frac{v}{5 + v + 0.02v^2}$$

Remember that this is a maximisation problem.

on the interval $[0, 35]$ and the corresponding value of v. Do this by carrying out the three steps of the Optimisation Procedure, just as you did in Activity 5.3(d).

(d) State your answer in a form which is appropriate in the context of the traffic planning model.

Solutions to parts (c) and (d) are given on page 29.

Comment

Having defined the function $f(v)$ for the volume flow rate of traffic in your worksheet, you may wish to display a graph of it. You can do so by defining a suitable graph range, for example, $v := 0, 0.1 \,..\, 35$, and then plotting $f(v)$ against v. While the graph confirms the maximum value at $v \simeq 16$, it also shows that there is little difference in the value of $f(v)$ between $v = 10$ and $v = 20$.

Save your file, for example as my121C1-traffic. Then close this file.

Chapter C2, Section 5
Integration with the computer

There is only one prepared Mathcad file for this section, and that comes towards the end and is optional. As in the later computer activities for Chapter C1, you will, for the most part, be creating your own worksheets here and using Mathcad directly. Subsection 5.1 shows how to find indefinite integrals in Mathcad, while Subsection 5.2 covers definite integrals.

5.1 Finding indefinite integrals

In Mathcad, indefinite integrals are found using the $\int$ operator. Like the $\frac{d}{dx}$ operator, the $\int$ operator can be used with Mathcad's symbolic commands, to find an algebraic expression for an integral of a given function.

In this subsection you are invited to find indefinite integrals for a variety of functions. The first activity provides an introduction to finding indefinite integrals using Mathcad.

Activity 5.1 How to find indefinite integrals

In this activity you will use Mathcad to find the indefinite integral of x^2.

The buttons referred to below are on the 'Calculus' and 'Symbolic' toolbars. If you wish to use these and they are not already visible, then either click on the appropriate buttons on the 'Math' toolbar, or select the **View** menu, **Toolbars** and choose **Calculus**, then repeat and choose **Symbolic**.

(a) Create a new (Normal) worksheet.

If necessary, see Chapter C1, Activity 5.3(a) on page 9 of this computer book.

(b) Enter the $\int$ operator in your worksheet, either by clicking on the $\int$ button on the 'Calculus' toolbar, or by using the keyboard alternative `[Ctrl]i`.

Be careful not to confuse the $\int$ button with the $\int_a^b$ button, which is used for finding definite integrals (as you will see later).

(c) Enter the expression to be integrated, x^2, in the first placeholder after the integral sign. (This expression is called the integrand.) Then enter the variable of integration, x, in the placeholder after the 'd'.

(d) Click on the $\rightarrow$ button ('Symbolic Evaluation') on the 'Symbolic' toolbar, or use `[Ctrl].`, the keyboard alternative. Then click elsewhere on the page, or press `[Enter]`, to obtain the integral. Check that the answer provided by Mathcad is what you expect.

(e) Now go through the same procedure to evaluate the integral $\int u^2\,du$.

(f) If you wish to save your work, then select the **File** menu and use **Save As...** to name and save your worksheet. (It is a good idea to insert a title in your worksheet. If you need to create space for this, then do so by positioning the red cross cursor at the top of the worksheet and pressing `[Enter]` to insert as many blank lines as required.)

Comment

◇ Notice that the $\int$ operator in Mathcad gives only *an* integral of the integrand. It does not give *the indefinite* integral because it does not add an arbitrary constant.

◇ The outcomes from integrating $\int x^2\,dx$ and $\int u^2\,du$ demonstrate that the form of the indefinite integral depends on the nature of the function being integrated but not on the choice of symbol for the variable of integration.

Mathcad notes

◇ In part (e), it is sufficient simply to edit the integral expression $\int x^2\,dx$, replacing each 'x' by a 'u'.

◇ You can also find an integral of a function f by first defining f and then evaluating symbolically the expression $\int f(x)\,dx$.

◇ The behaviour of the $\int$ operator differs somewhat from that of the $\frac{d}{dx}$ operator, used in Chapter C1. The result of evaluating the $\int$ operator symbolically is *not* affected by a value being defined for the integration variable (x, say) earlier in the worksheet. Furthermore, the $\int$ operator *cannot* be evaluated numerically, since this makes no sense in the context of finding an indefinite integral. If you try this, then Mathcad responds with the word 'function' to the right of the equals sign.

In each of the remaining activities in this subsection, you can either continue using the worksheet from the previous activity, or close that file and create a new worksheet. You will need to decide whether you wish to save your work.

In the next activity, you are asked to use Mathcad to find the indefinite integrals of two functions that you integrated by hand in the main text, and of a third function which you could also integrate by hand.

Activity 5.2 Finding indefinite integrals

If necessary, refer to the instructions in Activity 5.1(b)–(d).

If you arrive at an incorrect answer to part (c), then see the Mathcad notes below.

Use Mathcad to find each of the following indefinite integrals.

(a) $\displaystyle\int \left(\frac{1}{x} + e^{3x}\right) dx \quad (x > 0)$ (b) $\displaystyle\int \left(\frac{3}{y^4} + 5\sin(5y)\right) dy \quad (y > 0)$

(c) $\displaystyle\int (a + \cos(ax))\,dx$ (where a is a non-zero constant)

Solutions are given on page 29.

Comment

◇ After the '$+c$' has been added, the Mathcad answers to parts (a) and (b) agree with those obtained in the main text, and that for part (c) is identical to the answer found by applying Table 1.1. In part (a), recall that $\exp(3x)$ is an alternative way of writing e^{3x}.

◇ Note that, while Mathcad gives answers here that agree with those obtained by hand, there is no simple way of entering the additional constraints on the variables that accompany the integrals, for example, the condition $x > 0$ in part (a). For this reason, it is necessary to bear in mind that the symbolic manipulations performed by Mathcad might not be valid in all circumstances, and that the output should be interpreted with care.

Mathcad notes

It is safest to enter all products explicitly in Mathcad, by clicking on the 'Multiplication' × button on the 'Calculator' toolbar or by typing `*`. If you enter `3x` in part (a), or `5sin` or `5y` in part (b), then Mathcad will assume that you intended to enter a product, and will insert the multiplication for you. However, Mathcad will *not* help in this way if you enter `ax` rather than `a*x` in part (c). In this case, Mathcad assumes that you have entered a single variable name, 'ax', rather than the product of the variables a and x.

The next activity contains four integrals which you found by hand in Section 2, followed by three further integrals (in parts (e)–(g)) that cannot be found by hand simply on the basis of what is given in the chapter.

Activity 5.3 Further indefinite integrals

Use Mathcad to find each of the following indefinite integrals.

(a) $\int (x-3)(x-1)\,dx$ (b) $\int \frac{2x-3}{\sqrt{x}}\,dx$

(c) $\int \sin^2 x\,dx$ (d) $\int \frac{x}{x^2+1}\,dx$

(e) $\int ue^{3u}\,du$ (f) $\int x^2\ln(5x)\,dx$ (g) $\int \frac{1}{\sqrt{9-t^2}}\,dt$

Note that $\sin^2 x$ should be input as $\sin(x)^2$; for example, type `sin(x)^2`.

Solutions are given on page 30.

Comment

- ◇ The answers obtained from Mathcad in parts (a)–(d) are equivalent to the answers found by hand. However, they are not always given in an identical form, and in part (c) some algebra is required to show the equivalence of the two forms.
- ◇ It is possible to use the $\frac{d}{dx}$ operator symbolically to differentiate each of the integrals obtained in this activity. This leads to an expression equivalent to the original integrand in each case, as expected. However, the output obtained is not always identical in form to that of the original integrand, and again some algebra is sometimes required to show the equivalence of the two forms.
- ◇ Recall (from the context of obtaining derivatives symbolically) that the symbolic keywords 'simplify' and 'factor' can be used as alternatives to →. When applied to integrals, either of these may again provide an outcome in a more convenient form.

See the final Comment item for Chapter C1, Activity 5.1 on page 6 of this computer book.

Parts (e)–(g) of Activity 5.3 show that Mathcad can extend your 'integration reach' beyond the types of functions that you have so far learnt how to integrate by hand. However, Mathcad cannot integrate every function. For example, if you ask Mathcad to evaluate symbolically either of the indefinite integrals

$$\int e^{-t^3}\,dt \quad \text{and} \quad \int \sqrt{\sin^3 x}\,dx,$$

No expression in terms of standard functions is known for either of these integrals.

the response is for Mathcad to repeat the given integral without alteration (except for the replacement of e^{-t^3} by $\exp(-t^3)$ in the first case, and the square root by power $\frac{1}{2}$ in the second). This is how Mathcad responds when it cannot find an integral.

5.2 Definite integrals, areas and summations

You will need Mathcad file 121C2-01 for (optional) Activity 5.8, later in this subsection. First, however, you are invited to create your own Mathcad worksheets as in the previous subsection, but now to find definite rather than indefinite integrals.

In Mathcad definite integrals are found using the $\int_a^b$ operator. This operator can be used either symbolically or numerically. If the operator is used *symbolically*, then Mathcad finds an algebraic expression for an integral, evaluates this expression at the upper and lower limits of integration, and subtracts the second value from the first to find the answer. This is the same as the usual approach to finding definite integrals by hand. If the operator is used *numerically*, then Mathcad does not find an algebraic expression, but instead uses a numerical algorithm to find an approximate value for the definite integral.

In each of Activities 5.4–5.7 below, as in Subsection 5.1, you can either continue using the worksheet from the activity before, or close that file and create a new worksheet. You will need to decide whether you wish to save your work.

Activity 5.4 introduces you to the symbolic use of the $\int_a^b$ operator.

Activity 5.4 How to evaluate definite integrals

In this activity you will use Mathcad to evaluate the definite integral

$$\int_2^3 \frac{1}{x}\,dx.$$

(a) Enter the $\int_a^b$ operator in your worksheet, either by clicking on the $\int_a^b$ button on the 'Calculus' toolbar, or by using the keyboard alternative `&` (the ampersand sign, given on the keyboard by `[Shift]7`).

The `[Tab]` key provides a good way of moving around the placeholders.

(b) Enter the integrand, the variable of integration, and the upper and lower limits of integration in the appropriate placeholders.

(c) Click on the → button ('Symbolic Evaluation') on the 'Symbolic' toolbar, or use `[Ctrl].`, the keyboard alternative. Then click elsewhere on the page, or press `[Enter]`. You should obtain the answer

$$\ln(3) - \ln(2).$$

(d) Click anywhere on this answer, and then enter `=`, either by clicking on the `=` button on the 'Calculator' toolbar or by typing `=`, to evaluate it numerically. You should obtain the answer 0.405, which is the value of $\ln 3 - \ln 2$ to three decimal places.

Comment

Evaluating symbolically an expression involving the $\int_a^b$ operator gives an expression which is an *exact* answer (unless the original expression contains a decimal point; see the second Mathcad note below). In the example in this activity the expression is $\ln(3) - \ln(2)$. You can display the decimal value of such an expression by evaluating it numerically. This is done by selecting the expression, and then entering `=`.

Mathcad notes

◇ When you evaluate numerically an expression in Mathcad, the number of decimal places displayed is determined by the value of 'Number of decimal places'. The default value of this is 3, but you can change it by choosing **Result...** from the **Format** menu and then the 'Number Format' tab.

◇ If a Mathcad expression involving the $\int_a^b$ operator has a decimal point in any constant in the integrand, or in *both* limits of integration, then evaluating the expression symbolically gives a decimal answer with up to 20 decimal places. (Such an answer is unaffected by the value of 'Number of decimal places'.) For example, evaluating symbolically the integral below in Mathcad gives the outcome

$$\int_2^3 \frac{1.0}{x}\,dx \;\rightarrow\; .40546510810816438198.$$

The next activity provides further practice in using Mathcad to evaluate definite integrals symbolically.

Activity 5.5 Evaluating definite integrals

Each of parts (a)–(d) below gives a definite integral that you were asked to evaluate by hand in the main text. In each case, use Mathcad to evaluate the definite integral symbolically to obtain an exact answer, and then evaluate this answer numerically, to display it as a decimal value.

If necessary, refer to the instructions in Activity 5.4.

(a) $\displaystyle\int_0^2 e^t\,dt$ (b) $\displaystyle\int_0^{\pi/4} (\cos(5x) + 2\sin(5x))\,dx$

(c) $\displaystyle\int_1^2 \frac{6}{u^2}\,du$ (d) $\displaystyle\int_0^{\pi} e^t \sin t\,dt$

Remember that π can be obtained from the 'Calculator' or 'Greek' toolbar, or by typing `[Ctrl][Shift]p`.

Solutions are given on page 30.

In the next activity you will see an example of a definite integral that Mathcad is unable to evaluate symbolically. When this happens it is often worth attempting to evaluate the integral numerically, and the activity shows you how to do this.

Activity 5.6 An awkward definite integral

(a) Use Mathcad to try to evaluate symbolically the definite integral

$$\int_0^1 e^{-t^3}\,dt.$$

You should find that the definite integral is repeated without alteration (except for the replacement of e^{-t^3} by $\exp(-t^3)$). This means that Mathcad has been unable to calculate an algebraic expression for an integral of the integrand, and so it cannot evaluate symbolically the given definite integral.

At the end of Subsection 5.1 it was pointed out that Mathcad cannot evaluate the indefinite integral

$$\int e^{-t^3}\,dt.$$

(b) Evaluate the definite integral numerically, as follows. Click anywere on the expression just created, and then enter `=`.

You should find that the answer 0.808 is displayed.

Comment

- To evaluate any definite integral numerically, you should enter it in the same way as for symbolic evaluation, then select it and enter =.
- The reason why some definite integrals can be evaluated numerically but not symbolically in Mathcad is that symbolic evaluation requires Mathcad to find an algebraic expression for an integral, whereas numerical evaluation involves the use of a numerical algorithm. The answer obtained from this algorithm is an approximation, though usually an accurate one.

Activity 5.8 indicates a possible basis for such an algorithm.

Mathcad notes

On rare occasions, the numerical method used by Mathcad for evaluating definite integrals fails to produce a value. In such a case, the integral is marked with the error message 'Can't converge to a solution.'.

The next activity gives you further practice in using the $\int_a^b$ operator numerically.

Activity 5.7 Evaluating definite integrals numerically

The first two steps here are identical to those for symbolic integration. These steps would also have been required for the definite integral in Activity 5.6(b), had the expression not previously been entered.

(a) Evaluate numerically the definite integral $\displaystyle\int_0^1 t^2\,dt$, as follows.

(i) Either click on the $\int_a^b$ button on the 'Calculus' toolbar, or type & to insert the definite integral operator and its placeholders.

(ii) Fill in the four placeholders appropriately.

(iii) The blue editing lines should be on the expression; if not, select some part of the integral. Then evaluate the expression numerically, by entering =.

(b) Evaluate numerically the definite integral $\displaystyle\int_{-1}^1 \sqrt{1-x^2}\,dx$.

Solutions are given on page 30.

Comment

These two definite integrals can be evaluated either numerically or symbolically. As you can check, symbolic evaluation gives $\frac{1}{3}$ for (a) and $\frac{1}{2}\pi$ for (b).

You have seen that in Mathcad many definite integrals can be evaluated either symbolically or numerically. You might wonder which it is more appropriate to invoke in any given situation. If you want an exact answer (so you can see where constants such as π feature in it, for example), or if a general result is required, such as a formula for

$$\int_0^1 \cos(kx)\,dx \quad \text{(where } k \text{ is a non-zero constant),}$$

then use the symbolic approach. If you simply want a number that is an accurate value for the definite integral, then numerical evaluation should suffice. If you want both an exact value and a decimal value for the answer, then you can first evaluate the definite integral symbolically and afterwards enter =.

For example, the values of all the definite integrals in Activity 5.5 can be found satisfactorily using numerical integration.

Integration as a limit of summations

In Subsection 4.2, you saw that a definite integral could be approximated as closely as required by a finite sum. This was demonstrated in particular for the case in which the value of the definite integral gives the area beneath the graph of a function, and each finite sum represents the overall area of a set of rectangles. Each rectangle is based upon a subinterval, and the approximation to the definite integral improves as the number of subintervals is increased.

The remainder of this subsection will not be assessed.

For the area beneath the graph of the function $f(x)$ between $x = a$ and $x = b$, which is given exactly by the definite integral

$$\int_a^b f(x)\,dx,$$

We assume (as in the main text) that $f(x)$ takes only non-negative values in the interval $[a, b]$.

the approximation based on N subintervals is

$$\sum_{i=0}^{N-1} hf(a+ih) = h\sum_{i=0}^{N-1} f(a+ih), \quad \text{where } h = \frac{b-a}{N}.$$

See equation (4.1) in Chapter C2, Subsection 4.2.

The remaining (optional) activity in this section asks you to use the prepared Mathcad file 121C2-01 to explore the relationship between these approximations to the area under the curve and the definite integral itself.

Activity 5.8 Integration as a limit of summations (Optional)

(a) Open Mathcad file **121C2-01 Integration as a limit of summations**, and read through the worksheet, which consists of a single page. The definite integral being approximated here is

$$\int_0^{40} 15\sqrt{\sin\left(\frac{\pi x}{40}\right)}\,dx,$$

whose value was sought by a process of successive approximation in Subsection 4.2.

The value of this integral is the area on the graph beneath the red curve. The value calculated for A is an estimate for this area, using approximation by rectangles based on N subintervals. The value of N is initially set to 4.

Investigate the effect of increasing the number of subintervals, N. Use in turn the following values for N:

20, 50, 100, 500, 1000, 5000, 10 000.

Compare the corresponding values obtained for A with those in the right-hand column of the table below.

Number of subintervals	Sum of areas of rectangles
4	402.27
20	452.71
50	456.41
100	457.21
500	457.62
1 000	457.64
5 000	457.66
10 000	457.66

This is Table 4.1 in Chapter C2, Subsection 4.2.

(b) You may like to use the worksheet for other definite integrals, to investigate the behaviour of area estimates A as N is increased. For example, you could investigate these estimates for the definite integral

$$\int_0^1 e^{-x^3}\,dx,$$

by first editing the worksheet so that $f(x) = e^{-x^3}$ and $b = 1$.

If you have time, look also at the behaviour of the corresponding area estimates for the definite integral

$$\int_0^{\pi/4} \tan x\,dx.$$

Comment

◇ You should find that the numerical values obtained for A in part (a) match to two decimal places the values given in the right-hand column of the table. These area estimates appear to tend to the limit 457.66 (to two decimal places), which is also the numerical value provided by Mathcad for the definite integral.

◇ The values of the definite integrals suggested in part (b) are

$$\int_0^1 e^{-x^3}\,dx = 0.808 \quad \text{and} \quad \int_0^{\pi/4} \tan x\,dx = 0.347,$$

each to three decimal places. In each case, the estimates A converge to this value as N increases.

In the first example, the area estimates are greater than the value given by the definite integral for the area under the curve, while in the second example they are smaller. You can see this illustrated on the graph by setting small values of N ($N \leq 20$, say). In the first example, the tops of the rectangles all lie above the curve, while in the second example they all lie below.

Mathcad notes

◇ The summation sign is obtained from the $\sum_{n=1}^{m}$ button on the 'Calculus' toolbar, or by typing `[Ctrl]$` (for which you have to press the three keys `[Ctrl]`, `[Shift]` and 4 together).

◇ Note that the definite integral is set up in this file with integrand $f(x)$, where the function $f(x)$ is defined earlier in the worksheet. This approach is possible for either numerical or symbolic evaluation of a definite integral.

◇ The rectangles are filled in by drawing zig-zag lines very close together. On the screen these lines give the appearance of a solid block of colour, but the tiny gaps between the lines may become apparent if printed.

Now close Mathcad file 121C2-01.

Chapter C3, Section 4
Differential equations with the computer

In this section you will use the computer to see how the information contained in a differential equation can be displayed graphically, and how differential equations can be 'solved' numerically and graphically, even where no formula for the solution can be found.

There are two prepared Mathcad files associated with this section. The first draws direction fields and plots solution curves (graphs of solutions) for first-order differential equations, and the second illustrates how a numerical method for obtaining the solution to an initial-value problem works in practice.

This numerical method can be applied to any differential equation of the form

$$\frac{dy}{dx} = f(x, y),$$

where f is a known function of two variables. This form includes the types of differential equation considered in the main text, but also others.

As special cases, f may be a function of x alone or of y alone.

Mathcad does not contain any symbolic facilities for solving differential equations directly, to find a formula for the solution. However, as you saw in Chapter C2, Mathcad can be used to find integrals, and finding integrals is the main constituent of the two methods for solving first-order differential equations that are introduced in the main text (direct integration and separation of variables). Hence Mathcad can be used indirectly to help solve some first-order differential equations. Mathcad can also be applied to differentiate a solution that you have already found, to check that it is indeed a solution to the given differential equation.

Differentiation was the subject of Chapter C1.

In this section, however, Mathcad will be used numerically and graphically rather than symbolically.

4.1 Direction fields and solution curves

The differential equation

$$\frac{dy}{dx} = x + y$$

cannot be solved, using the methods of this chapter, to give an equation relating x and y. However, the direction field of this differential equation provides enough information to indicate the different types of solution curve that occur. In Activity 4.1 you will see this direction field drawn by Mathcad, and will be able to ask Mathcad to plot the solution curve through any point of your choice.

Activity 4.1 The differential equation $dy/dx = x + y$

Open Mathcad file **121C3-01 Direction fields and solution curves**. The worksheet opens with the direction field of the differential equation

$$\frac{dy}{dx} = x + y$$

drawn for a grid of points with integer coordinates in the (x, y)-plane, for values of x between -5 and 5, and values of y between -2 and 4. For each such point (x, y) a line segment is plotted through the point, and the slope of this line segment is $f(x, y) = x + y$.

The initial condition is $y(x_0) = y_0$.

(a) Set S to 1, so that a solution curve is plotted through the point $(0, 0)$. Briefly describe the curve obtained, or make a small sketch of it. Now change the value of y_0, to obtain a solution curve through each of the following points in turn:

$$(0, 1), \quad (0, 2), \quad (0, -1), \quad (0, -2).$$

In each case, note down a brief description of the curve or make a small sketch of it.

(b) Can you group the solution curves seen in part (a) into distinct types of behaviour? How many different types of behaviour are there?

For each of the following points, try to predict from looking at the direction field which type of behaviour the solution curve through the point will exhibit:

$$(-3, -1), \quad (-1, 0), \quad (4, 2).$$

Then, by making a suitable choice of values for x_0 and y_0, use Mathcad to plot the corresponding solution curve and to confirm your prediction.

Solutions are given on page 30.

Comment

The line segments of the direction field give a good indication of where the solution curves lie.

Mathcad notes

◇ A definition for a function of two variables is created in the same way as that for one variable. For example, to define the function $f(x, y) = x + y$ in Mathcad, you could type `f(x,y):x+y`.

◇ The calculations used to draw the direction field and solution curve are 'hidden' off the page of the Mathcad worksheet to the right. The area beyond the right-hand margin of a Mathcad page (which is marked by a solid vertical line) can be used just like the rest of the worksheet. It is divided into further pages, where you can place mathematical expressions, text, graphs and pictures. You do not need to look at the calculations in this worksheet, but in general, you can view what is in the 'hidden' area by using the horizontal scroll bar to move to the right.

In the next activity you are asked to use Mathcad to plot the direction fields for two other first-order differential equations. In each case, you can try to visualise from the direction field how the solution curves behave, before plotting some of these curves.

Activity 4.2 Direction fields and solution curves

You should still be working with Mathcad file 121C3-01.

(a) Investigate the direction field and solution curves for the first-order differential equation

$$\frac{dy}{dx} = e^{\cos x} - 1,$$

as follows.

(i) First set the variable S to zero, so that no solution curve is plotted. Then enter the right-hand side for this differential equation into the definition for $f(x, y)$.

(ii) Try to predict from the direction field where the solution curves lie. What types of behaviour will the solution curves exhibit?

(iii) Set the variable S to 1 to display a solution curve, and then try different values for x_0 and y_0 to confirm the predicted behaviour of solution curves. Use your observations to describe the behaviour of the solution curves for the differential equation.

(b) Now follow the same procedure as in part (a) to investigate the direction field and solution curves for the first-order differential equation

$$\frac{dy}{dx} = xy + 1.$$

However, in this case, start by altering the scope of the grid in the y-direction, by setting $Y1 := -5$, $Y2 := 5$ and $q := 10$.

Solutions are given on page 31.

Comment

◇ The differential equation in part (a) is of the form $dy/dx = f(x)$, and so can be solved in principle by direct integration. However, it turns out not to be possible to find an algebraic expression for

$$\int (e^{\cos x} - 1)\, dx.$$

The presence of the $\cos x$ means that the slopes of the direction field repeat at horizontal intervals of 2π. (The slopes are also invariant in the vertical direction, for any given choice of x, as noted in the solution.) The alternate positive and negative slopes indicate that solution curves will undulate. The overall increasing trend is not so obvious from the direction field, but might be expected because the magnitude (steepness) of the slopes appears to be greater where the slopes are positive than where they are negative.

◇ The differential equation in part (b) cannot be solved by the methods of this course. Using Mathcad to plot the direction field provides a good approach to visualising the types of solution curve for such a differential equation.

Now close Mathcad file 121C3-01.

4.2 Euler's method

In Subsection 4.1 you saw how a direction field can be used to visualise the information provided by a first-order differential equation. You also saw solution curves plotted on top of the direction field. It is straightforward to plot such curves for a first-order differential equation whose solutions can be expressed in terms of a simple algebraic formula. However, even where no such formula can be found, approximate solution curves for a differential equation can still be plotted. These graphs are based on a sequence of numerical estimates for solution values, which constitute a numerical solution to the differential equation. In this subsection you will see how such a numerical solution can be obtained.

This was the case for each of the differential equations in Activity 4.2.

The procedure to be described below, for obtaining an approximate numerical solution to a first-order differential equation, is known as *Euler's method*. To see how the solution is built up step by step, it is illuminating to consider the corresponding graphical construction. This involves forming a connected chain of line segments, each of which has a gradient given by the slope of the direction field at the left-hand end of the line segment, as shown in Figure 4.1.

Recall that a direction field provides a slope value at every point within a given region, and not just at the particular grid points displayed on a graph.

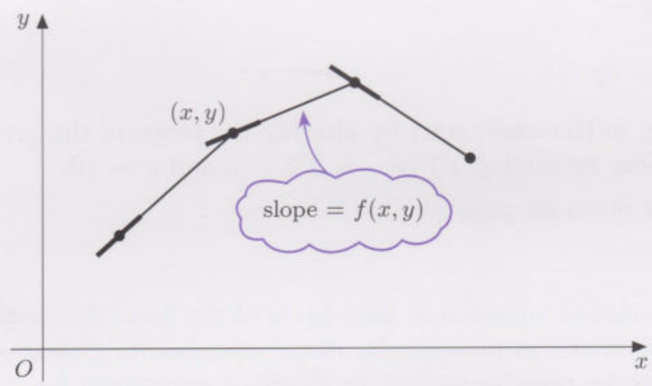


Figure 4.1 Graphical construction for Euler's method, where $dy/dx = f(x, y)$

You will see this idea explained in greater detail in the next activity.

Activity 4.3 Introducing Euler's method

Open Mathcad file **121C3-02 Euler's method**, read the introduction and turn to page 2 of the worksheet. Here a direction field has been set up for the function $f(x, y) = e^{\cos x} - 1$, that is, for the differential equation

$$\frac{dy}{dx} = e^{\cos x} - 1.$$

You studied this differential equation in Activity 4.2(a).

Also, an initial condition is specified, as

$$y = 0 \text{ when } x = 0; \quad \text{that is,} \quad y(0) = 0.$$

In the file, the initial values of x and y are denoted respectively by x_0 and y_0.

Note, to the right of the graph, that the value $f(x_n, y_n) = 1.718$ is displayed, where $n = 0$. This is (to 3 d.p.) the value of

$$f(0, 0) = e^{\cos 0} - 1 = e - 1,$$

which is the slope of the direction field at the starting point $(x_0, y_0) = (0, 0)$. This starting point is denoted on the direction field by a small blue box, and the direction of the direction field at this point is coloured magenta.

You will now see how an approximate solution to this initial-value problem can be built up graphically, step by step. Ensure that all of page 2 from the heading 'Solution curve' to the bottom of the graph is visible on your screen.

(a) Change the value under 'Number of steps' in turn to $N = 1$, $N = 2$, $N = 3$, $N = 4$ and $N = 5$. In each case, observe the effects that the change in value causes.

(b) Observe the effect of changing, in turn, the step size to $h = 0.5$ and the number of steps to $N = 10$.

(c) Observe the effect of changing the initial values. For example, set $x_0 = 1$ and $y_0 = 2$.

Comment

◇ When 'Number of steps' is changed to $N = 1$, the first segment of the approximate solution curve is drawn, from the starting point $(x_0, y_0) = (0, 0)$ to the point $(x_1, y_1) = (1, 1.718)$. The coordinates of these points appear in the tables to the right of the graph. The slopes of the direction field at these points, $f(x_0, y_0) = 1.718$ and $f(x_1, y_1) = 0.717$, are also shown.

Comment on part (a) for $N = 1$

All of the values are shown to three decimal places. The slope of the direction field at (x_1, y_1) is now shown by a line segment on the graph, coloured magenta.

The two broken blue line segments (one coinciding with the x-axis) that appear on the graph are temporary construction lines. They illustrate how the first segment of the solution curve is drawn as the hypotenuse of a right-angled triangle. The base (run) of this triangle is equal to the *step size*, which is specified above the graph as $h = 1$. The triangle is constructed, as shown in Figure 4.2, so that *the gradient of its hypotenuse is equal to the gradient given by the direction field at its left-hand vertex.* This gradient is $f(x_0, y_0)$. On the other hand, the gradient (slope) of the hypotenuse is given as usual by the rule 'slope equals rise over run', where the run is the step size, h. The rise, which is the height of the triangle, is therefore given by

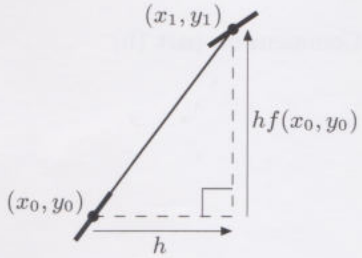


Figure 4.2 First segment

$$\text{rise} = \text{run} \times \text{slope} = hf(x_0, y_0).$$

The coordinates (x_1, y_1) of the top vertex of the triangle can then be calculated from the coordinates (x_0, y_0), the rise and the run. This gives

$$\begin{aligned}(x_1, y_1) &= (x_0 + \text{run},\ y_0 + \text{rise}) \\ &= (x_0 + h,\ y_0 + hf(x_0, y_0)) \\ &= (0 + 1,\ 0 + 1.718) = (1,\ 1.718).\end{aligned}$$

◇ When the number of steps is changed to $N = 2$, the second segment of the approximate solution curve is added, from the point $(x_1, y_1) = (1, 1.718)$ to the point $(x_2, y_2) = (2, 2.435)$. The broken blue lines now illustrate the 'construction triangle' for this second line segment, as shown also in Figure 4.3. The base (run) of this triangle is again equal to the step size, $h = 1$, but the gradient of its hypotenuse is equal to the gradient given by the direction field at (x_1, y_1). This gradient is $f(x_1, y_1) = 0.717$, as indicated by the table on the right of the screen. The coordinates (x_2, y_2) are therefore given by

Comment on part (a) for $N = 2$

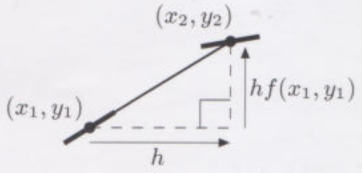


Figure 4.3 Second segment

$$\begin{aligned}(x_2, y_2) &= (x_1 + \text{run},\ y_1 + \text{rise}) \\ &= (x_1 + h,\ y_1 + hf(x_1, y_1)) \\ &= (1 + 1,\ 1.718 + 0.717) = (2,\ 2.435).\end{aligned}$$

Comment on part (a) for $N = 3$, $N = 4$ and $N = 5$

Note that the values of $f(x_n, y_n)$ given in the table are negative for $n = 2, 3$ and 4. These correspond to the negative slopes of the direction field at the corresponding points (x_n, y_n), as is clear from the graph.

◇ With two steps now completed, it should be fairly clear how the process continues when the number of steps is changed to $N = 3$, $N = 4$ and $N = 5$. The chain of line segments (which form the approximate solution curve) and the rows of corresponding values in the tables to the right of the graph build up one by one as N is incremented. Each time, the latest line segment is joined to the previous one, and its slope matches that of the direction field at its left-hand end.

Mathematically, this means that in constructing the line segment from (x_n, y_n) to (x_{n+1}, y_{n+1}), we have

$$\begin{aligned}(x_{n+1}, y_{n+1}) &= (x_n + \text{run}, y_n + \text{rise})\\ &= (x_n + h, y_n + hf(x_n, y_n)) \quad (n = 0, 1, 2, \ldots).\end{aligned}$$

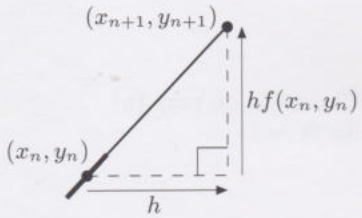


Figure 4.4 $(n + 1)$th segment

This is illustrated in Figure 4.4. In other words, the sequences x_n and y_n are determined by the pair of recurrence relations

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + hf(x_n, y_n) \quad (n = 0, 1, 2, \ldots),$$

as shown to the right of the graph on screen, together with the specified values for x_0 and y_0. These recurrence relations encapsulate **Euler's method**.

Each value y_n is an estimate of the 'true solution' y at $x = x_n$; that is, y_n is an estimate of $y(x_n)$. Clearly, the sequence of estimates obtained depends on the choice of both the step size, h, and the number of steps, N.

Comment on part (b)

◇ With N steps of size h, Euler's method provides $N + 1$ solution estimates, spaced at regular horizontal intervals between the chosen starting value, x_0, and $x_0 + Nh$.

Note that, when starting at $(x_0, y_0) = (0, 0)$, either 5 steps of size 1 or 10 steps of size 0.5 provide a final solution estimate at $x = 5$. By halving the step size and doubling the number N of steps, their product Nh, and hence also $x_0 + Nh$, remains the same. Notice, however, that the solution estimates obtained at $x = 5$ are different: the final coordinates are $(x_5, y_5) = (5, 0.986)$ in the first case and $(x_{10}, y_{10}) = (5, 0.558)$ in the second. Halving the step size and doubling the number of steps improves the accuracy of the solution estimate. In the next activity you can investigate further how accuracy improves with reductions in step size.

Comment on part (c)

◇ Setting $x_0 = 1$ and $y_0 = 2$ produces a different chain of line segments, which is again an approximate solution curve for the differential equation, now with the initial condition $y(1) = 2$. (The same is true for other choices of initial values.) If h and N are unaltered from part (b), then the last row of values at the right of the screen corresponds to a point which lies outside the displayed graph region.

Activity 4.3 showed how Euler's method works. You saw a connected chain of line segments built up, one by one, as an approximation to a solution curve. However, this was a fairly inaccurate approximation. The construction utilises slope values provided by the direction field only at points which are a horizontal distance h apart, where the values assigned to the step size h in Activity 4.3 were first 1 and then 0.5. As a result, the approximate solution curve was based on very limited information.

More information can be extracted from the direction field by reducing the step size, provided that also the number N of steps is increased to maintain coverage of the x-values over which a solution is sought. The next activity shows that such use of extra information leads to improvements in accuracy, and that an estimate for the solution $y(x)$ at a particular chosen value of x can, in principle, be found to whatever accuracy is required.

Activity 4.4 Using Euler's method

Turn to page 3 of the worksheet, and read the first paragraph. Then scroll down until all of the page from the heading 'Graphical and numerical solution' to the bottom of the graph is visible on your screen.

You should still be working with Mathcad file 121C3-02.

The page is set up to solve the same initial-value problem that was considered in Activity 4.3(a) and (b), namely,

$$\frac{dy}{dx} = e^{\cos x} - 1, \quad y(0) = 0.$$

The computation, using Euler's method, takes place from the starting value $x = x_0$ to the finishing value $x = xval$, where $xval$ can be specified by the user. The step size h can be chosen as before. However, in contrast to the situation in Activity 4.3, it is not possible here to vary independently the number of steps, N. Given a step size h, the number of steps to be used is calculated automatically from the condition that the final step is to reach $x = xval$. As a result, the value calculated for y_N is always an estimate for the value $y(xval)$ of the true solution at $x = xval$.

In other words, if h divides exactly into $xval - x_0$, then $N = (xval - x_0)/h$.

(a) Change the value of $xval$ to 9 (from its starting value of 5). What changes on the screen, and what stays the same?

(b) We now seek an estimate for the value of $y(9)$ (the true solution value at $x = 9$). This is obtained by progressively reducing the step size.

First note the value of y_N that appears on screen (this is the estimate for $y(9)$ obtained with step size $h = 1$, and hence with 9 steps).

Now change the step size in turn to 0.5, 0.2, 0.1, 0.01, 0.001 and 0.0001. In each case, note the corresponding value for y_N. Use these values to estimate the value of $y(9)$ to one decimal place.

Solutions are given on page 31.

Comment

◇ As the step size h is decreased, the number of steps increases. With $h = 0.0001$, there are 90 000 steps, for which the calculation may take an appreciable time on your computer.

◇ With $h = 0.5$, the approximate solution curve still looks like a connected chain of line segments. However, with $h = 0.2$ the graph appears significantly more like a smooth curve, and this remains the case for smaller values of h. (The graph is still in fact made up of short line segments, but so is every graph drawn by Mathcad with the trace type set to 'lines'!)

◇ Just as the numerical estimates for $y(xval)$ appear to converge as the step size is decreased, so too do the approximate solution curves seem to converge towards a 'limiting curve' on the direction field. The graphs obtained provide increasingly close representations of the actual solution curve.

Euler's method, as illustrated in Activities 4.3 and 4.4, was used to plot the solution curves in file 121C3-01, used for Activities 4.1 and 4.2. There, the number of steps was set to $N = 1000$, and calculations were made to draw the line segments to the left, as well as to the right, of the initial point. In that way, the line segments gave the appearance of a smooth solution curve over the whole horizontal graph range.

If you are interested and have the time, then you might like to try the following optional activity, which involves a function $f(x, y)$ that depends on the dependent variable y as well as on the independent variable x. For this particular example, it is also possible to check the outcome against a formula for the solution.

Activity 4.5 Another initial-value problem (Optional)

You should still be working with Mathcad file 121C3-02, on page 3 of the worksheet. Before doing anything else, reset the step size to $h = 1$ (see the first item in the Comment below).

(a) Use Euler's method to estimate to three decimal places the value of $y(6)$, where $y(x)$ satisfies the initial-value problem

$$\frac{dy}{dx} = x - y - 3, \quad y(1) = 1.$$

(For the step size h, use in turn the values 1, 0.5, 0.1, 0.01, 0.001 and 0.0001.)

(b) Check that the function $y = x - 4 + 4e^{1-x}$ is the solution to the initial-value problem in part (a). Hence find $y(6)$ exactly. Does this agree with the value that you obtained using Euler's method in part (a)?

Solutions are given on page 32.

Comment

See the Comment for Activity 4.4 of Chapter B1, in Computer Book B.

- ◇ If you do not start by setting $h = 1$, then any other change will cause recalculation for the most recently-used value of the step size, $h = 0.0001$, which may be time-consuming. Alternatively, recall that any Mathcad calculation can be interrupted by pressing `[Esc]` and then clicking 'OK' in the resulting option box. You may prefer to change here to 'manual calculation mode'.
- ◇ There is no need to alter any part of the worksheet before page 3, nor to change the parameters which define the grid for the direction field. On page 3 you need to alter the direction field function definition to $f(x, y) = x - y - 3$. Also, the values of x_0 and y_0 should both be set to 1, and the value of $xval$ to 6.
- ◇ Using the suggested values of h in turn, the numerical estimates y_N appear to converge and the approximate solution curves do likewise. The 'chain of line segments' is visible for $h = 1$ and for $h = 0.5$, but no departures from smoothness are apparent on the graph for smaller step sizes.

Now close Mathcad file 121C3-02.

Solutions to Activities

Chapter C1

Solution 5.1

Where more than one expression is given below for a solution, the first is similar to the Mathcad output and the second is a form that you are more likely to obtain by hand. (You found each of the derivatives by hand in the main text of Chapter C1, as indicated by the references below.)

(a) $3x^2 - 12x - 15$

See Activity 2.2(a).

(b) $4\pi r^2$

See Activity 3.4(a).

(c) $\dfrac{(\cos(t) - 2t)}{\exp(t)} - \dfrac{(\sin(t) - t^2)}{\exp(t)}$

$= \dfrac{\cos t - \sin t + t^2 - 2t}{e^t}$

See Exercise 4.2(b).

(d) $2\cos(x^2)x = 2x\cos(x^2)$

See pages 48–49.

(e) $-4\sin(4x)$

See Activity 4.7(a).

(f) $2t\ln(t) + \dfrac{(t^2 + 3)}{t}$

See Activity 4.2(c).

(g) $\dfrac{1}{u(u^2+3)} - 2\dfrac{\ln(u)}{(u^2+3)^2}u = \dfrac{u^2 + 3 - 2u^2\ln u}{u(u^2+3)^2}$

See Activity 4.4(b)

(h) $\dfrac{(\exp(t) + 1)}{(\exp(t) + t)} = \dfrac{e^t + 1}{e^t + t}$

See Exercise 4.3(d).

Solution 5.3

(d) *Step 1:* The one stationary point of the function

$$f(x) = 0.125\sqrt{1 + x^2} + 0.0625(2 - x),$$

with domain $[0, 2]$, is at $x = 0.577$ (to 3 d.p.).

Step 2: The values of f at the interval endpoints are

$$f(0) = 0.250 \quad \text{and} \quad f(2) = 0.280,$$

while the value of f at the stationary point is

$$f(0.577) = 0.233 \quad \text{(all to 3 d.p.)}.$$

Step 3: Hence the minimum value of $f(x)$ for x in the interval $[0, 2]$ is 0.233 at $x = 0.577$ (both to 3 d.p.).

Hence the solution to the orienteer's problem is to aim to join the path at a distance 0.577 km from the fixed point O on the path.

(This agrees with the answer 0.58 km found in Activity 5.7(a) of Chapter A3 in Computer Book A.)

Solution 5.4

(c) *Step 1:* The stationary points of the function

$$f(v) = \frac{v}{5 + v + 0.02v^2}$$

are at $v = \pm 15.811$ (to 3 d.p.).

Step 2: Only the positive stationary point, 15.811, lies inside the interval $[0, 35]$. The values of f at the interval endpoints are

$$f(0) = 0 \quad \text{and} \quad f(35) = 0.543,$$

while the value of f at the stationary point in the interval is

$$f(15.811) = 0.613 \text{ (all to 3 d.p.)}.$$

Step 3: Hence the maximum value of $f(v)$ for v in the interval $[0, 35]$ is 0.613 at $v = 15.811$ (both to 3 d.p.).

(d) According to the model, a maximum traffic flow rate of about 0.61 vehicles per second can be achieved, by keeping the speed of traffic at about 16 $\mathrm{m\,s^{-1}}$ (that is, about 57 km per hour or 35 mph).

Chapter C2

Solution 5.2

In each case, c is an arbitrary constant which has been added to the expression given by Mathcad. The answers to parts (a) and (b) agree with those obtained earlier by hand.

(a) $\displaystyle\int \left(\frac{1}{x} + e^{3x}\right) dx = \ln(x) + \tfrac{1}{3}\exp(3x) + c$

See Activity 1.2(a).

(b) $\displaystyle\int \left(\frac{3}{y^4} + 5\sin(5y)\right) dy = -\frac{1}{y^3} - \cos(5y) + c$

See Exercise 1.1(b).

(c) $\displaystyle\int (a + \cos(ax))\, dx = ax + \frac{\sin(ax)}{a} + c$

Solution 5.3

In each case, c is an arbitrary constant which has been added to the expression given by Mathcad. The Mathcad answers in parts (a), (b) and (d) resemble closely those obtained by hand in the main text.

(a) $\displaystyle\int (x-3)(x-1)\,dx = \tfrac{1}{3}x^3 - 2x^2 + 3x + c$

See Example 2.1(a).

(b) $\displaystyle\int \frac{2x-3}{\sqrt{x}}\,dx = \tfrac{4}{3}x^{3/2} - 6x^{1/2} + c$

See Example 2.1(c).

(c) $\displaystyle\int \sin^2 x\,dx = -\tfrac{1}{2}\cos(x)\sin(x) + \tfrac{1}{2}x + c$

In Activity 2.3(a), the answer $\frac{1}{2}x - \frac{1}{4}\sin(2x) + c$ was obtained by hand. The equivalence of this and the Mathcad expression follows from the double-angle formula $\sin(2x) = 2\sin x\cos x$.

(d) $\displaystyle\int \frac{x}{x^2+1}\,dx = \tfrac{1}{2}\ln(x^2+1) + c$

See Activity 2.5(b)(iii).

(e) $\displaystyle\int ue^{3u}\,du = \tfrac{1}{3}u\exp(3u) - \tfrac{1}{9}\exp(3u) + c$

(f) $\displaystyle\int x^2\ln(5x)\,dx = \tfrac{1}{3}x^3\ln(5x) - \tfrac{1}{9}x^3 + c$

(g) $\displaystyle\int \frac{1}{\sqrt{9-t^2}}\,dt = \arcsin(\tfrac{1}{3}t) + c$

(The function arcsin is represented in Mathcad by 'asin'.)

Solution 5.5

The answers are presented in a form as close as possible to the Mathcad output (which by default is given to 3 decimal places). Each answer agrees with that obtained by hand in the main text.

(a) $\displaystyle\int_0^2 e^t\,dt = \exp(2) - 1 = 6.389$

See Activity 4.4(b).

(b) $$\int_0^{\pi/4} (\cos(5x) + 2\sin(5x))\,dx = \tfrac{1}{10}2^{1/2} + \tfrac{2}{5} = 0.541$$

See Exercise 4.1(a).

(c) $\displaystyle\int_1^2 \frac{6}{u^2}\,du = 3$

See Exercise 4.1(b). (This answer is the Mathcad outcome from symbolic evaluation.)

(d) $\displaystyle\int_0^{\pi} e^t\sin t\,dt = \tfrac{1}{2}\exp(\pi) + \tfrac{1}{2} = 12.070$

See Exercise 4.1(c).

Solution 5.7

The answers are given by Mathcad to 3 decimal places.

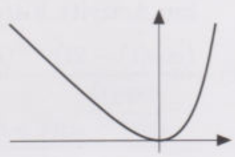
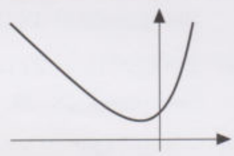
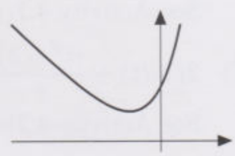
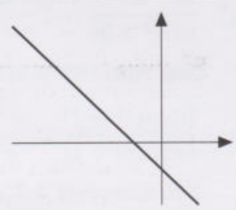
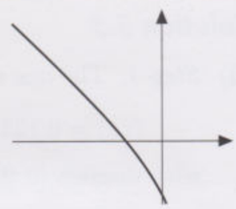
(a) $\displaystyle\int_0^1 t^2\,dt = 0.333$

(b) $\displaystyle\int_{-1}^1 \sqrt{1-x^2}\,dx = 1.571$

Chapter C3

Solution 4.1

(a) The solution curves are described and sketched in the following table.

(x_0, y_0)	Description	Sketch
$(0,0)$	Rough U-shape; steeper to right of minimum than to left.	
$(0,1)$	Similar, but with minimum to the left and higher.	
$(0,2)$	Similar, but with minimum still further to the left and higher.	
$(0,-1)$	Straight line $y=-x-1$.	
$(0,-2)$	Downward curve with gradient decreasing.	

(b) The solution curves show three distinct types of behaviour:

(i) Any solution curve through a point above the line $y=-x-1$ remains above that line. It decreases to a minimum and then increases.

(ii) The line $y=-x-1$ is itself the solution curve through any point on that line.

(iii) Any solution curve through a point below the line $y = -x - 1$ remains below that line and decreases.

The solution curves through the points $(-3, -1)$, $(-1, 0)$ and $(4, 2)$ are of types (iii), (ii) and (i), respectively.

Solution 4.2

(a) (ii) The slope of the direction field at a point (x_0, y_0) appears to depend on the choice of x_0 alone and not on that of y_0. Correspondingly, there will be solution curves of just one type, with any two such curves differing only by a vertical translation. (This is a consequence of the fact that the function $f(x, y)$ in this case depends only on x.)

(iii) All choices for x_0 and y_0 give a solution curve which increases and decreases alternately, but has a rising trend. Each such curve may be obtained from that shown below by a vertical translation.

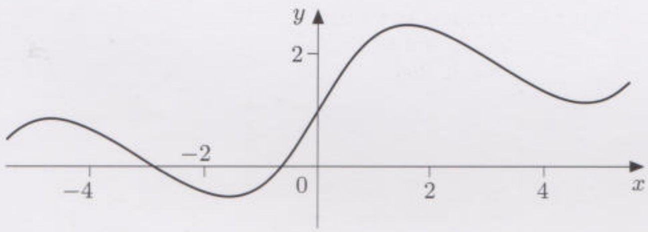


Figure S3.1

(b) (ii) The effect of the direction field on solution curves in this case is not so clear-cut. However, it appears that there may be three types of solution curve, as detailed below.

(iii) There are solution curves that cross the x-axis and are increasing, as in the graph below. If $x_0 = 0$, then such curves are obtained by a choice of y_0 such that $-1.25 \leq y_0 \leq 1.25$.

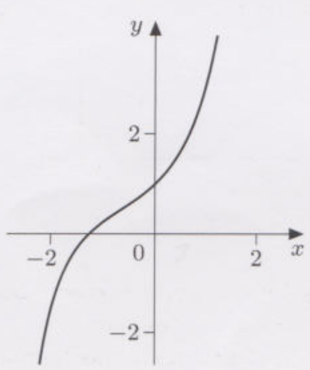


Figure S3.2

There are solution curves that lie above the x-axis, each having a minimum in the second quadrant, as in the graph below. If $x_0 = 0$, then such curves are obtained by a choice of y_0 such that $y_0 \geq 1.26$.

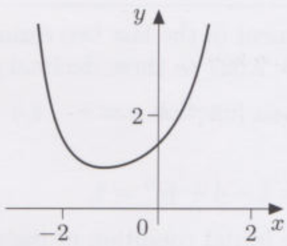


Figure S3.3

There are solution curves that lie below the x-axis, each having a maximum in the fourth quadrant, as in the graph below. If $x_0 = 0$, then such curves are obtained by a choice of y_0 such that $y_0 \leq -1.26$.

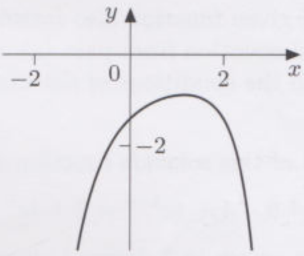


Figure S3.4

Solution 4.4

(a) The graph extends to $x = 9$, but the portion to the left of $x = 5$ remains unchanged. In particular, the step size h does not alter, though the number N of steps does (so that the graph reaches $x = 9$ rather than $x = 5$).

(b) A table of the values obtained is below.

h	1	0.5	0.2	0.1	0.01	0.001	0.0001
y_N	3.916	3.347	3.002	2.887	2.783	2.772	2.771

The values of y_N appear to be converging and, given the level of agreement between the last two estimates, it seems reasonable to deduce from them an estimate for $y(9)$ that is accurate to one decimal place, that is, $y(9) = 2.8$. (In fact, it looks likely that $y(9) = 2.77$ to 2 decimal places.)

Solution 4.5

(a) The table below gives the values obtained using Euler's method, from file 121C3-02.

h	1	0.5	0.1	0.01	0.001	0.0001
y_N	2.000	2.004	2.021	2.026	2.027	2.027

The agreement of the last two estimates suggests that $y(6) = 2.027$ to three decimal places.

(b) For the given function $y = x - 4 + 4e^{1-x}$, we have

$$y(1) = 1 - 4 + 4e^0 = 1,$$

so that the initial condition of the problem in part (a) is satisfied.

The derivative of the given function is

$$\frac{dy}{dx} = 1 - 4e^{1-x},$$

whereas we have

$$\begin{aligned} x - y - 3 &= x - (x - 4 + 4e^{1-x}) - 3 \\ &= 1 - 4e^{1-x}. \end{aligned}$$

Hence the given function also satisfies the differential equation from part (a), and therefore satisfies all the conditions of the initial-value problem.

The value of this solution function at $x = 6$ is

$$y(6) = 6 - 4 + 4e^{1-6} = 2 + 4e^{-5} \simeq 2.027.$$

This value agrees to 3 decimal places with that obtained using Euler's method.

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